

# Wigner Glass, Spin-liquids, and the Metal-Insulator Transition

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Recent experiments on the two dimensional ( $d = 2$ ) electron gas in various semiconductor devices have revealed an unexpected metal-insulator transition [1–4] and have challenged the previously held assumption that there is no such transition in two dimensions. While the experiments are still at the stage of rapid development, it is becoming evident that they cannot be understood from the conventional perspective of weak interactions. In the present paper, we propose the following. (1) The low-density insulating state is the Wigner Glass, a phase with quasi-long-range translational order and competing ferromagnetic and antiferromagnetic spin-exchange interactions. (2) The transition is the melting of this Wigner Glass, disorder being the agent allowing the transition to be second order. (3) Within the Wigner Glass phase, there are at least two, distinct magnetic ground-states, a ferromagnetic state at very low electron density and a spin-liquid state with a spin pseudo-gap at higher densities. (4) The metallic side of the transition is a non-Fermi liquid. These conclusions are encapsulated in Figure 1 which presents the proposed phase diagram as a function of disorder strength and density; we also suggest experimental signatures of the various phases and transitions.

Although there is not complete agreement between experiments, a number of striking features have emerged. Most importantly, the temperature dependence of the resistivity changes at a critical carrier concentration ( $n_c$ ): for lower carrier concentration ( $n < n_c$ ), it is insulating, implying a diverging resistivity as the temperature,  $T \rightarrow 0$ ; for higher carrier concentration ( $n > n_c$ ), it is metallic, the resistivity decreasing with decreasing temperature. Therefore,  $n_c$  appears to signify a zero temperature quantum phase transition between two fundamentally distinct states of matter. It was previously thought that such a metal-insulator transition is not possible in two dimensions [5]. Indeed, previous experiments, carried out over the past two decades on systems with higher electron density where the effects of electron-electron interactions are significantly weaker, strongly supported the idea that a metallic state in two dimensions is not possible.

In some of the recent experiments, a striking reflection

symmetry is observed [6]. It is striking because, *a priori*, there is no particular symmetry between the two distinct states of matter, the metal and the insulator. Over a significant range of  $n$  around  $n_c$ , one finds that  $g^*(\delta_n, T) = 1/g^*(-\delta_n, T)$ , where  $g^*$  is the conductance scaled by the value at the critical concentration, which is of order  $e^2/h$ ,  $e$  being the electronic charge and  $h$  the Planck's constant, and  $\delta_n = (n - n_c)/n_c$ . Concomitantly, the data obeys scaling, that is, the data can be collapsed onto two universal curves, one for the metallic samples and one for the insulating samples, when it is plotted against the scaling variable  $[T_0(\delta_n)/T]^{1/x}$ , where  $x$  is a critical exponent, and the scaling parameter is  $T_0(\delta_n) \sim |\delta_n|^x$ . The resistivity as a function of the electric field shows a similar scaling behavior [7], and the current-voltage characteristic shows equally striking reflection symmetry; the data on the insulating side can be mapped on to the metallic side by a simple reflection.

The immediate conclusion one can draw from these experiments is that the scaling and reflection properties are due to a correlation length diverging at a quantum critical point [8] corresponding to a quantum phase transition. Moreover, the observation that the range of carrier concentration over which reflection symmetry is obeyed shrinks with decreasing temperature is consistent with a zero temperature quantum critical point [9]. Therefore, although the phenomenological scaling theory [8] appears to be valid, the following key questions remain unaddressed: *What is the origin of the quantum critical point? What is the nature of insulating and conducting phases?* In this paper we substantively address the origin of the quantum critical point and the nature of the insulating phases, and offer a conjecture regarding the conducting phase.

There have been theoretical attempts to explain the experiments in terms of a conventional Fermi liquid approach [10] pioneered by Finkel'stein [11]. At best, such a perturbative renormalization group approach can provide hints concerning the existence of a quantum critical point, since this point lies outside the perturbative regime [12]. In addition, the nature of the strong coupling phases discussed below are such that they simply cannot be accessed in perturbation theory. Because the experiments do not indicate a finite superconducting transition temperature in samples with  $n > n_c$ , it remains to be seen if the “metallic phase” is actually a superconductor in

disguise [13–15]. The sensitivity of the resistivity data to magnetic fields at the temperatures probed in these experiments [2,16] appears inconsistent with any classical explanation.

New insight into this problem of the disordered two dimensional electron gas can be obtained if it is approached from the strong interaction (low density) limit [17]; the requisite quantum critical point is due to the zero temperature continuous melting of a new state of matter, called the Wigner *glass*, which, as shown below, is distinct from either an Anderson insulator, or a Mott insulator [5]. At this quantum critical point scaling must hold. In contrast, in the absence of disorder, the insulating triangular Wigner crystal, which minimizes the Coulomb energy, quantum melts via a first order transition at which scaling does not apply. Thus, the situation is radically different with and without disorder. It is the principal purpose of this paper to analyze the nature and the consequences of the existence of the Wigner glass.

The presence of electronic spin degrees of freedom distinguishes the Wigner glass from the similar Bragg glass [18] which has been discussed in other contexts. The spin degrees of freedom lead to a magnetic Hamiltonian which is highly frustrated due to the presence of significant ring-exchange processes. Frustration, in turn, can lead to novel magnetic phases, including a “spin-liquid” [19], where a “spin-liquid” is a state in which quantum fluctuations are so strong that even at  $T = 0$  no symmetries are broken. As the Wigner glass melts, the electrons are no longer power-law ordered at the lattice sites, but the local crystalline character and the short distance antiferromagnetic spin-singlet correlations should survive within the scale of the correlation length, which is large close to the quantum critical point. Most likely such a state cannot be a Fermi liquid due to its strong singlet correlations. Moreover, the observed conducting state cannot be a Fermi liquid for it would localize if it were [8]. A simple model of a non-Fermi liquid has a perfect metal ground state if the interactions are sufficiently strong [20], and the interactions are certainly strong in the present experiments. We conjecture that the conducting state is a non-Fermi liquid metal with strong spin singlet correlations.

For a Wigner glass, in the presence of disorder, it follows from the work of Giamarchi and Le Doussal [18] that the equal-time correlation function

$$C_G(R) = \overline{\langle e^{i\mathbf{G}\cdot\mathbf{u}(\mathbf{R})} e^{-i\mathbf{G}\cdot\mathbf{u}(0)} \rangle} \sim \frac{1}{R^\eta}, \quad R \rightarrow \infty, \quad (1)$$

where  $\mathbf{u}(\mathbf{R}, t)$  is the displacement at the lattice site  $\mathbf{R}$ , and  $\eta$  is a critical exponent. The angular brackets denote the groundstate average and the overline the disorder average. Therefore, the system exhibits quasi-long-range crystalline order characterized by power law Bragg peaks at the reciprocal lattice vectors,  $\mathbf{G}$ , of the triangular lattice, instead of the  $\delta$ -function peaks of a crystal with true

long-range order.

This quasi-long-range crystalline order is possible only if unbound dislocations are not generated by disorder. Giamarchi and Le Doussal [18] have given a self-consistent argument that this is indeed the case for a range of parameters, but, in  $d = 2$ , this is still an open question; for  $d = 3$ , this idea has received further support, recently [21,22]. Pending further investigation, we shall assume the existence of the Bragg glass phase in  $d = 2$ . Even if the self-consistent argument of Ref. [18] turns out to be incorrect, dislocations are expected to be exponentially rare [22] and so of not great practical importance. On the other hand, at very low carrier concentrations, corresponding to very high disorder, dislocations will proliferate, destroying this phase, and the resulting state is indistinguishable from the Mott-Anderson insulating phase. Thus, the Wigner glass should exist only at intermediate carrier concentrations. It is also worth noting that the dissipative properties of this glass at the melting point should be very different from the melting of the Bragg glass because of the continuum of low lying fermionic excitations.

A crucial question is whether or not the Wigner glass is a distinct state of matter. The findings of Giamarchi and Le Doussal imply that it is. Specifically, it is characterized by a power-law crystalline order (See, Eq. (1)) and by broken replica symmetry [23], as in a spin glass. As a consequence, there are many low lying metastable states, with barriers between them diverging as a function of the “separation” in the phase space. Such a state cannot be accessed by perturbative renormalization group methods [11].

As the density is increased, the Wigner glass will quantum melt in a continuous manner, in contrast to the first order melting of a Wigner crystal; in the presence of disorder, coexistence of phases is not possible, and, therefore, a first order transition is forbidden [24]. However, because replica symmetry is restored in the melted liquid, there must still be a phase transition and hence a quantum critical point!

The pure Wigner crystal transition in  $d = 2$  is believed to occur at  $r_s \approx 37$  [25], where  $r_s$  is a measure of the importance of electron-electron interactions and is defined as the average interparticle separation in units of the effective Bohr radius  $a_B = \hbar^2 \epsilon / m_b e^2$ ;  $\epsilon$  is the dielectric constant and  $m_b$  is the effective mass of the charge carriers. The situation changes dramatically in the presence of disorder, because the elastic Wigner glass can take advantage of pinning and lower its energy relative to the liquid phase. There are numerical results that indicate that the critical value of  $r_s$  can be lowered substantially, for example, to 7.5 from 37 in the presence of disorder [26]. The critical  $r_s$  is, of course, a nonuniversal quantity and can depend strongly on the precise microscopic model.

The local crystalline order of a Wigner glass allows us

to estimate exchange energies. For this purpose, it is legitimate to treat the glass as a triangular crystal, recognizing that the exchange constants will be randomly distributed. A general theorem shows that exchanges involving an even number of Fermions are antiferromagnetic, while those involving an odd number of Fermions are ferromagnetic [27,28]. There are estimates of these multiparticle exchanges in both solid  $^3\text{He}$  and the Wigner crystal [29]. The version of the many dimensional WKB [30] formalism used here to calculate them is different in detail from that of Ref. [29]. The actual potential at short distances is, of course, softer than the Coulomb potential,  $e^2/\epsilon r$ , due to the spread of the wavefunctions of the electrons perpendicular to the plane. However, we have explicitly verified that a more realistic softer potential makes a negligible difference (a little larger exchange energies) as long as the spread in the perpendicular direction is less than half the lattice spacing, which is the case by a safe margin. This implies that our results are independent of the short distance microscopic details and depend only on the nature of the potential on the scale of the lattice spacing, which typically is of order  $300\text{\AA}$ .

The semi-classical expression for a  $p$ -particle exchange energy  $J_p$  is

$$J_p = A_p \hbar \omega_0 \left( \frac{B_p}{2\pi\hbar} \right)^{1/2} e^{-B_p/\hbar}. \quad (2)$$

Here  $B_p$  is the value of the Euclidean action along the minimal action path that exchanges  $p$  electrons on a triangular lattice, and  $\omega_0$  is the attempt frequency, which can be estimated from the phonon spectrum of the Wigner lattice [31]. We ascribe the zero point energy of the phonons to an effective oscillator,  $\hbar\omega_0 = 1.63/r_s^{3/2}$  in units of the Rydberg constant, which is  $e^2/2\epsilon a_B$ . The prefactor  $A$ , in one dimensional WKB problems, can be proven to be greater than unity [33] and is often quite a bit larger than unity; this should hold more generally. We set  $A_p = 1$ , which is an underestimate. Equation (2) can be trusted as long as  $B_p/\hbar$  is much larger than unity.

We have calculated  $B_p/\hbar$  for a large number of exchange configurations (upto  $p = 18$ ) and have also made asymptotic estimates for  $p \gg 1$ . Here, we report results for only the most compact 2, 3, 4, 5, and 6-particle exchanges for the Coulomb potential, of which 5 and 6-particle exchanges were not previously calculated. This required minimization over typically 500-1000 variables to obtain satisfactory convergence. If we define  $B_p/\hbar = b_p r_s^{1/2}$ , then  $b_2 = 1.66$ ,  $b_3 = 1.52$ ,  $b_4 = 1.67$ ,  $b_5 = 1.91$ , and  $b_6 = 1.77$ . The values of  $b_p$  for  $p = 2, 3$ , and 4 are typically in the range 10-15% lower than the previously known estimates [29]; this is because full minimization was not previously carried out.

If we retain for simplicity only exchanges up to 4 particles, it is easy to show in terms of the spin-1/2 operators,

$\mathbf{S}$ , that for the perfect triangular lattice, the spin Hamiltonian is

$$H = \text{cst.} + J_{NN} \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + J_{NNN} \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + 4J_4 \sum_{\text{Rhombus}} G_{ijkl}, \quad (3)$$

where  $G_{ijkl} = (\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l) + (\mathbf{S}_i \cdot \mathbf{S}_l)(\mathbf{S}_j \cdot \mathbf{S}_k) - (\mathbf{S}_i \cdot \mathbf{S}_k)(\mathbf{S}_j \cdot \mathbf{S}_l)$ . The first sum is over all distinct nearest neighbors, where  $J_{NN} = 4(J_2 + \frac{5}{4}J_4 - J_3)$ . The second sum is over all distinct next nearest neighbors, where  $J_{NNN} = J_4$ . The third sum is over all distinct rhombuses.  $J_{NN}$  is antiferromagnetic for  $r_s \lesssim 39$ , and ferromagnetic for larger  $r_s$ . Its magnitude, using  $\epsilon = 7.7$  and  $m_b = 0.2$  (appropriate for Si-MOSFET) is 5.2 K, 1.6 K, 0.4 K, 0.06 K for  $r_s = 5, 7, 10$ , and 15, respectively. Note, however, that while  $J_{NNN}/J_{NN} \approx 0.4$  at  $r_s = 5$ , the ratio approaches 1 at about  $r_s = 29$ . This is a highly frustrated antiferromagnetic spin Hamiltonian. Numerical evidence [32] suggests that the ground state of this Hamiltonian is “quantum disordered” for  $r_s \lesssim 39$ , where “quantum disordered” means that spin-rotational symmetry is not broken, even at  $T = 0$ . Because the actual Wigner glass is not translationally invariant, any quantum disordered state is a spin liquid; various dimerized states, which break translational symmetry, are not distinct states of matter. However, because of disorder, the spin liquid only exhibits a spin pseudo-gap state, rather than the spin gap state of the pure system [32]. Other magnetic structures may develop for  $r_s \gtrsim 39$ , but they are likely to be predominantly ferromagnetic.

In the current experiments cited earlier, the critical values of  $r_s$  at which the metal-insulator transition takes place are in the range of 5-15. (The character of this transition will be very different if it takes place for larger values of  $r_s$  for which exchange is of insignificant importance at experimentally realistic temperatures.) Because the exchange energies at  $r_s \sim 10$  are about 0.4 K, the spin-singlets can be broken with modest magnetic fields of order 1T, and the ground state should be very sensitive to such fields. Once the spin-singlets are broken, the electrons can be described as a Fermi liquid, which should localize in  $d = 2$  due to disorder. Indeed, it has been observed that the metallic behavior in Si MOSFETs can be suppressed by such fields [16]. In the absence of the magnetic field, the continuous nature of the melting of the Wigner glass allows the spin correlations on the scale of the correlation length (large close to the melting transition) to be similar to those in the Wigner glass. In particular, the spin pseudogap character should survive the melting process, and the drop of the resistivity as a function of temperature seen in experiments is due to the reduction of the scattering rate of the carriers as the temperature is lowered due to the opening of the spin

pseudogap.

The qualitative global phase diagram at  $T = 0$  is sketched in Fig. 1 with  $D$  as the measure of the strength of the quenched disorder.

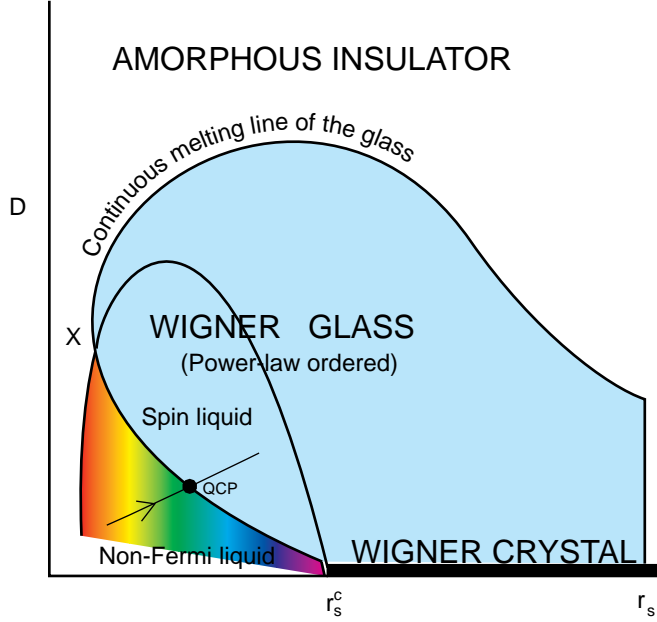


FIG. 1. The  $T = 0$  phase diagram as a function of disorder  $D$  and  $r_s$ . The point  $X$  is a tetracritical point. The arrowed curve represents the postulated experimental path; QCP is the corresponding quantum critical point. The Amorphous Insulator crosses over from an Anderson-Mott description at low  $r_s$  to Efros-Shklovskii behaviour at large  $r_s$ .

All the transition lines are continuous quantum phase transitions for which scaling holds. The point  $X$  is a tetracritical point, while  $r_s^c$  is the first order transition of the pure two-dimensional electron gas to a Wigner crystal at which scaling does not hold. In a Coulomb system, the density is determined by the charge-neutrality condition. Hence, the density is continuous; only the order parameter is discontinuous. The spin-liquid is a phase within the Wigner glass. The Fermi liquid behavior of the  $D = 0$  line is unstable with respect to arbitrarily weak disorder. It either flows towards the Mott-Anderson insulating state or to a non-Fermi liquid state. It is thus possible that the Mott-Anderson insulator to non-Fermi liquid phase boundary terminates in the  $D \rightarrow 0$  limit at the origin ( $D = r_s = 0$ ), or it may terminate at a point of finite interaction strength ( $D = 0$ ,  $r_s = r_s^A$ ). Also, note the reentrance of the Mott-Anderson insulator as a function of  $r_s$ .

The Wigner glass phase is stable to small thermal perturbations and must exhibit a finite temperature phase transition with a transition temperature that vanishes at the quantum critical point. However, neither the Mott-Anderson insulator, nor the non-Fermi liquid are distinct phases of matter at finite temperatures. Because of the continuous Heisenberg symmetry of the spin Hamilto-

nian, there cannot be any magnetic long range order in  $d = 2$  at any finite temperature. The transitions shown in the figure are true thermodynamic phase transitions at  $T = 0$ , and are, in principle, detectable as non-analyticities in thermodynamic functions, such as the chemical potential. A prediction is that the compressibility,  $\kappa$ , of the electrons will vanish as  $\kappa \sim (n - n_c)^\nu$ , which should be experimentally observable in capacitive measurements. (As defined earlier,  $\nu$  is the correlation length exponent that has been determined experimentally to be 1.5 from the temperature and the electric field scalings [7].) The argument is quite simple. From macroscopic electrodynamics, the dielectric function,  $\epsilon(q, \omega = 0) = 1 + \frac{2\pi e^2}{q} \kappa(q)$ . In the insulating Wigner glass state, the  $q = 0$  dielectric function is finite and different from that of the vacuum. This implies that  $\kappa(q) \sim q$  as  $q \rightarrow 0$ . Since  $\kappa(q, \xi)$  should be continuous across  $q\xi \sim 1$ , it follows from the scaling hypothesis that  $\kappa(q, \xi) \sim \frac{1}{\xi}$  in the critical regime,  $q\xi > 1$ . The critical behaviour of the compressibility should be observable in macroscopic measurements on the conducting side of the transition.

The magnetic transitions will be reflected in the magnetic response of the electron spins as  $T \rightarrow 0$ . In particular, a signature of the spin-liquid to ferromagnetic phase transition can be detected in the uniform susceptibility, which will diverge as the transition is approached from the spin-liquid side. A similar signature may be observable in the uniform susceptibility at the non-Fermi liquid to Mott-Anderson insulator transition. Because of the power-law crystalline order of the Wigner glass, its presence can also be detected in surface acoustic wave and narrow band noise measurements.

By applying electric field, a Wigner glass can be made to slide [34]. In general, there should not be any associated threshold field; although the equilibrium Wigner glass may not contain dislocations, these will be generated in the presence of an external electric field. The mechanism which destroys any possible threshold field is similar to that discussed by Coppersmith [35]. The current-voltage characteristic in the glass phase is  $J \sim E \exp(-(E_c/E)^\mu)$ , where  $E$  is the electric field,  $J$  is the current density,  $E_c$  is the *typical* pinning field, and  $\mu$  is an exponent. This reflects the presence of infinitely many low lying metastable states. As the glass melts, the current voltage characteristic should become linear at asymptotically low currents. Close to the transition, the scaling picture suggests that the conductivity can be written as  $\sigma(\omega, E, T) = \sigma \left( \frac{\delta_n}{\omega^{1/\nu z}}, \frac{\delta_n}{E^{1/\nu(z+1)}}, \frac{T}{\omega} \right)$ , where  $\nu z$  is precisely the exponent  $x$  defined earlier. The glass correlation length is given by  $\xi_g \sim \delta_n^{-\nu}$ , and  $z$  is the dynamic critical exponent defined by  $\xi_\tau \sim \xi_g^z$ , where  $\xi_\tau$  is the correlation time.

We emphasize, once again, that for purer systems, the metal-insulator transition must approach the first order

melting of a Wigner crystal at which scaling cannot hold, while in more disordered samples scaling will hold. The scaling theory [8] of the metal insulator transition involves a transition between an insulator and a perfect metal. This description is valid in the critical regime, even if the actual asymptotic behavior of the “metallic” phase is different from that of a perfect metal; outside the critical regime, the behavior of the “metallic” phase can be affected by couplings that are irrelevant at the critical fixed point (and so do not affect the critical phenomena), but are relevant at the putative stable fixed point. Thus, it is perfectly consistent with the scaling theory for the resistivity outside the critical regime to saturate at low temperatures, as has been observed in some experiments, or even to turn around and ultimately diverge at very low temperatures.

We hope that many aspects of our proposed phase diagram can be explored, both theoretically and experimentally, in the near future. It is also worthwhile to consider the extension of the present ideas to three dimensional systems, where the existence of a Wigner glass phase is on even firmer theoretical footing.

*Note added:* We thank V. M. Pudalov for drawing our attention to earlier evidence of a Wigner crystal transition in Si MOSFETs described in V. M. Pudalov, *et al.*, *Phys. Rev. Lett.* **70**, 1866 (1993). However, from the perspective of the present paper, this transition is better described as a Wigner glass transition with no threshold field.

After this paper was submitted for publication, further evidence of a Wigner glass transition was discovered (J. Yoon *et al.*, *cond-mat/9807235*). The authors describe this as a Wigner crystal transition from the proximity of the transition to  $r_s^c = 37$  obtained from fixed-node quantum Monte Carlo approximation. This interpretation is incorrect as the disorder is finite, which by all theoretical accounts must destroy the Wigner crystal. We believe that they have observed a transition to the Wigner glass state with a power-law order as described here, close, however, to the first order melting of the pure crystal. It is therefore not surprising that scaling is approximately violated; asymptotically close to the transition scaling must of course hold as the system is strictly speaking not pure.

H.-W. Jiang in private communications has reported to us that the compressibility in his hole doped GaAs samples indeed vanishes as a power-law at  $r_s^c \sim 8$  as described above.

The evidence of the transition to an insulating state at smaller values of  $r_s$  as shown in Figure 1 has been found (A. R. Hamilton *et al.*, *cond-mat/9808108*). The character of this second transition is very different from that of the first transition. This implies that there must be at least two distinct insulating states, as there cannot be two different critical transitions to the same insulating state from one conducting state. We suggest that this

lends further support to our phase diagram.

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